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Bravais Lattices in Four-dimensional Space

By A. L. Mackay

Birkbeck College Crystallographic Laboratory, 21, Torrington Square, London, W.C. 1, U.K.

and G. S. Pawley

Crystallographic Laboratory, Cavendish Laboratory, Cambridge, U.K.

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It is shown by enumeration that in four dimensions there are 52 translation groups (Bravais lattices). 21 of these are primitive and 31 are centred.

1. Introduction

S. I. Tomkeieff (1955) has stated without proof that in a space of n-dimensions denoted by $[n]$ there are

$$\sum_{k=1}^{n} k^2$$

space lattices, Bravais lattices or translation groups. He would thus expect there to be 30 lattices in [4].

Goursat (1889), Hermann (1949) and Hurley (1951) (the latter following Goursat's notation) deduced the 24 symmetry operations in [4] which do not include translation components. Hurley derived the 222 crystallographic point groups in [4], his list including the 122 groups found by Heesch who generated the black-and-white point groups in [3] regarding them as 'four-dimensional groups of three-dimensional space'.

Hermann (1949) has shown some ways in which n-dimensional crystallographic groups can be studied.

In section 3 it is shown that the papers of Goursat and Hurley are in agreement with that of Hermann as regards point-group symmetry operations and in section 4 onwards the [4] space lattices are developed and discussed.

52 groups are described.

2. Space lattices in $[n]$

The primitive lattices, each corresponding to an axial system, can be derived mathematically by the use of the metric tensor $[g_{ij}]$ where $i, j = 1, 2, 3, 4 \ldots n$. This tensor has been defined for [3] (*International Tables for X-ray Crystallography* II, p. 60 (1959)), n linearly independent translation vectors $\mathbf{a}_1, \mathbf{a}_2 \ldots \mathbf{a}_n$ define the unit cell. The terms g_{ij} of the metric tensor are the inner products of the unit-cell edges so that $g_{ij} = \mathbf{a}_i . \mathbf{a}_j$. This tensor is thus symmetrical and of the second order with $\frac{1}{2}n(n+1)$ independent elements. If the $\mathbf{a}_i$ transform to new axes by $\mathbf{a}'_i = T_{ij}\mathbf{a}_j$ then the g_{ij} transform by $g'_{ij} = T_{ik}T_{jl}g_{kl}$. Here and throughout this paper the Einstein summation convention is used. T_{ij} is used to represent a symmetry operation. To define a primitive lattice (or a coordinate system) only the independent components of $[g_{ij}]$ need be listed. The most general [4] lattice is thus represented by

$$[g_{ij}] = \begin{bmatrix} \mathbf{a}_1^2 & \mathbf{a}_1.\mathbf{a}_2 & \mathbf{a}_1.\mathbf{a}_3 & \mathbf{a}_1.\mathbf{a}_4 \\ & \mathbf{a}_2^2 & \mathbf{a}_2.\mathbf{a}_3 & \mathbf{a}_2.\mathbf{a}_4 \\ & & \mathbf{a}_3^2 & \mathbf{a}_3.\mathbf{a}_4 \\ & & & \mathbf{a}_4^2 \end{bmatrix}$$

which is only invariant under the symmetry operations I_{ij} (the identity operation) and $-I_{ij}$ (inversion in the origin). When there are more symmetry operations the number of independent components of $[g_{ij}]$ will be reduced, as the metric tensor must be invariant under the symmetry operations T_{ij} according to

$$g'_{ij} = T_{ik}T_{jl}g_{kl} = g_{ij} \; .$$

The properties that the content V of the unit cell is given by $V^2 = |g_{ij}|$ and the distance L corresponding to a translation $x_i\mathbf{a}_i$ is given by $L^2 = x_i x_j g_{ij}$ will be used later.

3. The [4] symmetry operations

If T_{ij} is one of the 24 point-group symmetry operations it can be represented by a 4 by 4 matrix which has the characteristic equation:

$$|\lambda I_{ij} - T_{ij}| = \lambda^4 - \chi\lambda^3 + \sigma\lambda^2 \pm \chi\lambda + |T_{ij}| = 0, \; |T_{ij}| = \mp 1$$

where χ, σ and $|T_{ij}|$ are functions of the coefficients of T_{ij}, and are different for each type of operation. $(\chi, \sigma, |T_{ij}|)$ can be used as a nomenclature for the operation (Hurley, 1951) to which Hurley also assigned an arbitrary letter. Hermann used symbols representing the multiplicities of the irreducible components D_{ij} of the symmetry operations. Thus if $(D_{ij})^m = I_{ij}$ where m is the smallest possible integer, then m is the multiplicity, and Hermann gives lists of the symbols for the various orders for each operation. If $|T_{ij}| = +1$ the operation is proper, if $|T_{ij}| = -1$ it is improper. $\chi = T_{ii}$ is the trace and σ is the second invariant of T_{ij}.

The remaining operations $I' = 2222$, $K' = 622$, $L' = X\,(=\text{ten})$, $M' = 34$, $N' = 321$, $R' = 422$, $S' = 33$, $T' =$

2221 and $Z'=322$ are formed from the corresponding unprimed operations above by changing the sign of the trace $\chi=T_{ii}$. Thus I' in Hurley's notation is identical to $-I_{ij}$ used previously. (See Table 1).

4. The [4] primitive lattices or axial systems

Many of the lattices in $[n]$ can be constructed simply by considering $[m]$ lattices orthogonal to $[n-m]$ lattices. Thus in [4] groups can be found which have two different plane groups at right angles and others which have a [3] group at right angles to a linear group. These cases can be seen from the form of $[g_{ij}]$. This approach, however, is inadequate for generating a complete set of [4] lattices, and it is necessary to examine how the 10 components of the metric tensor are reduced by the requirement that $[g_{ij}]$ should be invariant under the group of symmetry operations of a particular lattice. Each of the 24 symmetry operations restricts the components of $[g_{ij}]$, but operations do not occur in isolation but only in combination with other operations making up the point group. The condition that $[g_{ij}]$ should be invariant with respect to these other operations too, may restrict it further. The operations have, of course, to be described in appropriate orientations relative to each other. Each of the lattices will belong to one of Hurley's 222 point groups. The various metric tensors obtained by this procedure are listed in Table 2.

All the point groups containing $-I_{ij}$ should be treated in this way, which is an enormous task. In fact most of the lattices were found by the method

Table 1. *The* 24 *symmetry operations in* [4]

Multiplicity m	$(\chi, \sigma, \|T_{ij}\|)$	Hurley's letter	Hermann's symbol	Matrix
1	(4, 6, 1)	I	1111	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
8	(0, 0, 1)	A	8	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \bar{1} & 0 & 0 & 0 \end{pmatrix}$
6	(0, 1, 1)	B	63	$\begin{pmatrix} 0 & \bar{1} & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \bar{1} & \bar{1} \end{pmatrix}$
				*or $\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 1 & 0 & 0 \\ 0 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & -1 \end{pmatrix}$
12	(0, −1, 1)	C	T (=twelve)	$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \bar{1} & \bar{1} \\ 0 & \bar{1} & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$
				*or $\frac{1}{2}\begin{pmatrix} 0 & 0 & -1 & -\sqrt{3} \\ 0 & 0 & \sqrt{3} & -1 \\ 1 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 1 & 0 & 0 \end{pmatrix}$
4	(0, 2, 1)	D	44	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \bar{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \bar{1} & 0 \end{pmatrix}$
2	(0, −2, 1)	E	2211	$\begin{pmatrix} \bar{1} & 0 & 0 & 0 \\ 0 & \bar{1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
4	(0, 0, −1)	F	421	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \bar{1} & 0 & 0 & 0 \\ 0 & 0 & \bar{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
3	(1, 0, 1)	K	311	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ \bar{1} & \bar{1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
				*or $\frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} & 0 & 0 \\ \sqrt{3} & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

Table 1 (*cont.*)

5	(1, 1, 1)	L	5	$\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\\bar{1}&\bar{1}&\bar{1}&\bar{1}\end{pmatrix}$ *or $\frac{1}{4}\begin{pmatrix}1&1&3&\sqrt{5}\\1&-3&-1&\sqrt{5}\\-3&1&-1&\sqrt{5}\\-\sqrt{5}&-\sqrt{5}&\sqrt{5}&-1\end{pmatrix}$
12	(1, 2, 1)	M	64	$\begin{pmatrix}0&\bar{1}&0&0\\1&1&0&0\\0&0&0&\bar{1}\\0&0&1&0\end{pmatrix}$ *or $\frac{1}{2}\begin{pmatrix}1&\sqrt{3}&0&0\\-\sqrt{3}&1&0&0\\0&0&0&2\\0&0&-2&0\end{pmatrix}$
6	(1, 0, −1)	N	621	$\begin{pmatrix}0&\bar{1}&0&0\\1&1&0&0\\0&0&\bar{1}&0\\0&0&0&1\end{pmatrix}$ *or $\frac{1}{2}\begin{pmatrix}1&3&0&0\\-3&1&0&0\\0&0&-2&0\\0&0&0&2\end{pmatrix}$
4	(2, 2, 1)	R	411	$\begin{pmatrix}0&1&0&0\\\bar{1}&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
6	(2, 3, 1)	S	66	$\begin{pmatrix}0&\bar{1}&0&0\\1&1&0&0\\0&0&0&\bar{1}\\0&0&1&1\end{pmatrix}$ *or $\frac{1}{2}\begin{pmatrix}1&3&0&0\\-3&1&0&0\\0&0&1&3\\0&0&-3&1\end{pmatrix}$
2	(2, 0, −1)	T	2111	$\begin{pmatrix}\bar{1}&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$
6	(3, 4, 1)	Z	611	$\begin{pmatrix}0&\bar{1}&0&0\\1&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}$ *or $\frac{1}{2}\begin{pmatrix}1&3&0&0\\-3&1&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}$

* In general the components of T_{ij} can be integers in a suitable axial system but if orthogonal axes are used for describing rotations of 60° they become non-integral. These alternatives are given.

described at the beginning of this section and checked by the mathematical method. Only those groups with the irreducible symmetry operations A, C, L and L' had to be found by the mathematical enumerative method. It is due to this economy that omission is possible.

5. The non-primitive [4] lattices

Centred lattices have been enumerated by trial. Lattices are centred with the following restrictions:

(1) The point-group symmetry at each lattice point must be the same as in the uncentred lattice.

(2) Every lattice point must have the same environment.

(3) Halving of a translation merely gives the same lattice.

(4) The coordinates of the lattice points form a group closed with respect to addition and subtraction (mod. 1).

There are apparent exceptions to this procedure. In [3], for example, the rhombohedral lattice R is often described as a centring of the hexagonal P lattice. However, as the symmetry of the former is $\bar{3}m$ and of the latter $6/mmm$ the symmetry is lowered

Table 2

Serial number of lattice	Number of independent parameters	Order of group	Metric tensor	Unit cell edges in terms of orthogonal unit vectors **i, j, k, l**	Symmetry operations I, $-I$ and:
1	10	2	$\mathbf{a}_1^2 \quad \mathbf{a}_1\cdot\mathbf{a}_2 \quad \mathbf{a}_1\cdot\mathbf{a}_3 \quad \mathbf{a}_1\cdot\mathbf{a}_4$ $\mathbf{a}_2^2 \quad \mathbf{a}_2\cdot\mathbf{a}_3 \quad \mathbf{a}_2\cdot\mathbf{a}_4$ $\mathbf{a}_3^2 \quad \mathbf{a}_3\cdot\mathbf{a}_4$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_{21}\mathbf{i}+a_{22}\mathbf{j}$ $a_{31}\mathbf{i}+a_{32}\mathbf{j}+a_{33}\mathbf{k}$ $a_{41}\mathbf{i}+a_{42}\mathbf{j}+a_{43}\mathbf{k}+a_{44}\mathbf{l}$	
2	7	4	$\mathbf{a}_1^2 \quad 0 \quad 0 \quad 0$ $\mathbf{a}_2^2 \quad \mathbf{a}_2\cdot\mathbf{a}_3 \quad \mathbf{a}_2\cdot\mathbf{a}_4$ $\mathbf{a}_3^2 \quad \mathbf{a}_3\cdot\mathbf{a}_4$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_2\mathbf{j}$ $a_{32}\mathbf{j}+a_{33}\mathbf{k}$ $a_{42}\mathbf{j}+a_{43}\mathbf{k}+a_{44}\mathbf{l}$ [1] ⊥ [3] triclinic	$2T$
3	6	4	$\mathbf{a}_1^2 \quad \mathbf{a}_1\cdot\mathbf{a}_2 \quad 0 \quad 0$ $\mathbf{a}_2^2 \quad 0 \quad 0$ $\mathbf{a}_3^2 \quad \mathbf{a}_3\cdot\mathbf{a}_4$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_{21}\mathbf{i}+a_{22}\mathbf{j}$ $a_3\mathbf{k}$ $a_{43}\mathbf{k}+a_{44}\mathbf{l}$ [2] oblique ⊥ [2] oblique	$2E$
4	5	8	$\mathbf{a}_1^2 \quad 0 \quad 0 \quad 0$ $\mathbf{a}_2^2 \quad 0 \quad 0$ $\mathbf{a}_3^2 \quad \mathbf{a}_3\cdot\mathbf{a}_4$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_2\mathbf{j}$ $a_3\mathbf{k}$ $a_{43}\mathbf{k}+a_{44}\mathbf{l}$ [2] rectangular ⊥ [2] oblique	$2E$ $4T$
5	4	24	$\mathbf{a}_1^2 \quad -\frac{1}{2}\mathbf{a}_1^2 \quad 0 \quad 0$ $\mathbf{a}_1^2 \quad 0 \quad 0$ $\mathbf{a}_3^2 \quad \mathbf{a}_3\cdot\mathbf{a}_4$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_1(-\frac{1}{2}\mathbf{i}+(3^{\frac{1}{2}}/2)\mathbf{j})$ $a_3\mathbf{k}$ $a_{43}\mathbf{k}+a_{44}\mathbf{l}$ [2] hexagonal ⊥ [2] oblique	$2E$ $4K$ $12T$ $4Z$
6	4	16	$\mathbf{a}_1^2 \quad 0 \quad 0 \quad 0$ $\mathbf{a}_1^2 \quad 0 \quad 0$ $\mathbf{a}_3^2 \quad \mathbf{a}_3\cdot\mathbf{a}_4$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_1\mathbf{j}$ $a_3\mathbf{k}$ $a_{43}\mathbf{k}+a_{44}\mathbf{l}$ [2] square ⊥ [2] oblique	$2E$ $4R$ $8T$
7	4	16	$\mathbf{a}_1^2 \quad 0 \quad 0 \quad 0$ $\mathbf{a}_2^2 \quad 0 \quad 0$ $\mathbf{a}_3^2 \quad 0$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_2\mathbf{j}$ $a_3\mathbf{k}$ $a_4\mathbf{l}$ [2] rectangular ⊥ [2] rectangular	$6E$ $8T$
8	3	8	$\mathbf{a}_1^2 \quad \mathbf{a}_1\cdot\mathbf{a}_2 \quad 0 \quad 0$ $\mathbf{a}_2^2 \quad 0 \quad 0$ $\mathbf{a}_1^2 \quad \mathbf{a}_1\cdot\mathbf{a}_2$ $\mathbf{a}_2^2$	$a_1\mathbf{i}$ $a_{21}\mathbf{i}+a_{22}\mathbf{j}$ $a_1\mathbf{k}$ $a_{21}\mathbf{k}+a_{22}\mathbf{l}$ As no. 3 but equivalent	$2D$ $4E$
9	3	24	$\mathbf{a}_1^2 \quad 0 \quad 0 \quad 0$ $\mathbf{a}_2^2 \quad \alpha\mathbf{a}_2^2 \quad \alpha\mathbf{a}_2^2$ $\mathbf{a}_2^2 \quad \alpha_2^2$ $\mathbf{a}_2^2$	$a_1\mathbf{i}$ $a_2\mathbf{j}+a_4\mathbf{l}$ $a_2(-\frac{1}{2}\mathbf{j}+(3^{\frac{1}{2}}/2)\mathbf{k})+a_4\mathbf{l}$ $a_2(-\frac{1}{2}\mathbf{j}-(3^{\frac{1}{2}}/2)\mathbf{k})+a_4\mathbf{l}$ [1] ⊥ [3] rhombohedral	$6E$ $4K$ $4N$ $8T$
10	3	32	$\mathbf{a}_1^2 \quad 0 \quad 0 \quad 0$ $\mathbf{a}_1^2 \quad 0 \quad 0$ $\mathbf{a}_3^2 \quad 0$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_1\mathbf{j}$ $a_3\mathbf{k}$ $a_4\mathbf{l}$ [2] square ⊥ [2] rectangular	$10E$ $4F$ $4R$ $12T$
11	3	48	$\mathbf{a}_1^2 \quad -\frac{1}{2}\mathbf{a}_1^2 \quad 0 \quad 0$ $\mathbf{a}_1^2 \quad 0 \quad 0$ $\mathbf{a}_3^2 \quad 0$ $\mathbf{a}_4^2$	$a_1\mathbf{i}$ $a_1(-\frac{1}{2}\mathbf{i}+(3^{\frac{1}{2}}/2)\mathbf{j})$ $a_3\mathbf{k}$ $a_4\mathbf{l}$ [2] hexagonal ⊥ [2] rectangular	$14E$ $4K$ $8N$ $16T$ $4Z$

Table 2 (*cont.*)

Serial number of lattice	Number of independent parameters	Order of group	Metric tensor				Unit cell edges in terms of orthogonal unit vectors **i, j, k, l**	Symmetry operations I, $-I$ and:
12	2	96	$\mathbf{a}_1^2$	0	0	0	$a_1\mathbf{i}$	$18E$
				$\mathbf{a}_2^2$	0	0	$a_2\mathbf{j}$	$24F$
					$\mathbf{a}_2^2$	0	$a_2\mathbf{k}$	$16K$
						$\mathbf{a}_2^2$	$a_2\mathbf{l}$	$12R$
							[1] ⊥ [3] cubic	$24T$
13	2	96	$\mathbf{a}_1^2$	0	0	0	$a_1\mathbf{i}$	$26E$ $16N$
				$\mathbf{a}_1^2$	0	0	$a_1\mathbf{j}$	$12F$ $4R$
					$\mathbf{a}_3^2$	$-\frac{1}{2}\mathbf{a}_3^2$	$a_3\mathbf{k}$	$4K$ $20T$
						$\mathbf{a}_3^2$	$a_3(-\frac{1}{2}\mathbf{k}+(3^{1/2}/2)\mathbf{l})$	$8M$ $4Z$
							[2] square	
							⊥ [2] hexagonal	
14	2	64	$\mathbf{a}_1^2$	0	0	0	$a_1\mathbf{i}$	$4D$
				$\mathbf{a}_1^2$	0	0	$a_1\mathbf{j}$	$18E$
					$\mathbf{a}_3^2$	0	$a_3\mathbf{k}$	$16F$
						$\mathbf{a}_3^2$	$a_3\mathbf{l}$	$8R$
							[2] square	$16T$
							⊥ [2] square	
15	2	144	$\mathbf{a}_1^2$	$-\frac{1}{2}\mathbf{a}_1^2$	0	0	$a_1\mathbf{i}$	$8B$ $8S$
				$\mathbf{a}_1^2$	0	0	$a_1(-\frac{1}{2}\mathbf{i}+(3^{1/2}/2)\mathbf{j})$	$38E$ $24T$
					$\mathbf{a}_3^2$	$-\frac{1}{2}\mathbf{a}_3^2$	$a_3\mathbf{k}$	$8K$ $8Z$
						$\mathbf{a}_3^2$	$a_3(-\frac{1}{2}\mathbf{k}+(3^{1/2}/2)\mathbf{l})$	$48N$
							[2] hexagonal	
							⊥ [2] hexagonal	
16	2	48	$\mathbf{a}_1^2$	$\alpha\mathbf{a}_1^2$	$\alpha\mathbf{a}_1^2$	$\alpha\mathbf{a}_1^2$	$a_1(\mathbf{i}+\mathbf{j}+\mathbf{k}+\beta\mathbf{l})$	$12A$
				$\mathbf{a}_1^2$	$\alpha\mathbf{a}_1^2$	$\alpha\mathbf{a}_1^2$	$a_1(-\mathbf{i}-\mathbf{j}+\mathbf{k}+\beta\mathbf{l})$	$6D$
					$\mathbf{a}_1^2$	$\alpha\mathbf{a}_1^2$	$a_1(\mathbf{i}-\mathbf{j}-\mathbf{k}+\beta\mathbf{l})$	$12E$
						$\mathbf{a}_1^2$	$a_1(-\mathbf{i}+\mathbf{j}-\mathbf{k}+\beta\mathbf{l})$	$16K$
							[4] analogue of	
							[3] rhombohedral	
17	2	12	$\mathbf{a}_1^2$	$\frac{1}{2}\mathbf{a}_1^2$	0	$\alpha\mathbf{a}_1^2$	$\beta\mathbf{j}+\gamma\mathbf{k}$	$4C$
				$\mathbf{a}_1^2$	$-\alpha\mathbf{a}_1^2$	0	$-(3^{1/2}/2)\beta\mathbf{i}+\frac{1}{2}\beta\mathbf{j}+\frac{1}{2}\gamma\mathbf{k}+(3^{1/2}/2)\gamma\mathbf{l}$	$2D$
					$\mathbf{a}_1^2$	$\frac{1}{2}\mathbf{a}_1^2$	$-\gamma\mathbf{i}+\beta\mathbf{l}$	$4S$
						$\mathbf{a}_1^2$	$-\frac{1}{2}\gamma\mathbf{i}-(3^{1/2}/2)\gamma\mathbf{j}-(3^{1/2}/2)\beta\mathbf{k}+\frac{1}{2}\beta\mathbf{l}$	
18	1	1152	$\mathbf{a}_1^2$	$-\frac{1}{2}\mathbf{a}_1^2$	0	$\frac{1}{2}\mathbf{a}_1^2$	$\frac{1}{4}a_1(-\mathbf{i}+\mathbf{j}+\mathbf{k}+\mathbf{l})$	$144A$ $128K$
				$\mathbf{a}_1^2$	$-\frac{1}{2}\mathbf{a}_1^2$	0	$\frac{1}{4}a_1(\mathbf{i}-\mathbf{j}-\mathbf{k}+\mathbf{l})$	$96C$ $384N$
					$\mathbf{a}_1^2$	$-\frac{1}{2}\mathbf{a}_1^2$	$\frac{1}{4}a_1(\mathbf{i}+\mathbf{j}+\mathbf{k}-\mathbf{l})$	$12D$ $72R$
						$\mathbf{a}_1^2$	$\frac{1}{4}a_1(-\mathbf{i}-\mathbf{j}+\mathbf{k}+\mathbf{l})$	$90E$ $32S$
							24-cell lattice	$144F$ $48T$
19	1	288	$\mathbf{a}_1^2$	$-\frac{1}{2}\mathbf{a}_1^2$	0	0	$a_1\mathbf{i}$	$32B$ $8K$
				$\mathbf{a}_1^2$	0	0	$a_1(-\frac{1}{2}\mathbf{i}+(3^{1/2}/2)\mathbf{j})$	$24C$ $48N$
					$\mathbf{a}_1^2$	$-\frac{1}{2}\mathbf{a}_1^2$	$a_1\mathbf{k}$	$12D$ $8S$
						$\mathbf{a}_1^2$	$a_1(-\frac{1}{2}\mathbf{k}+(3^{1/2}/2)\mathbf{l})$	$50E$ $24T$
							as 15 but equivalent	$72F$ $8Z$
20	1	384	$\mathbf{a}_1^2$	0	0	0	$a_1\mathbf{i}$	$48A$ $64K$
				$\mathbf{a}_1^2$	0	0	$a_1\mathbf{j}$	$12D$ $64N$
					$\mathbf{a}_1^2$	0	$a_1\mathbf{k}$	$42E$ $24R$
						$\mathbf{a}_1^2$	$a_1\mathbf{l}$	$96F$ $32T$
							[4] analogue of	
							[3] cubic–hypercubic	
21	1	240	$\mathbf{a}_1^2$	$+\frac{1}{2}\mathbf{a}_1^2$	$+\frac{1}{2}\mathbf{a}_1^2$	$+\frac{1}{2}\mathbf{a}_1^2$	$(1/8^{1/2})a_1(\mathbf{i}+\mathbf{j}+\mathbf{k}-(5^{1/2})\mathbf{l})$	$30E$
				$\mathbf{a}_1^2$	$+\frac{1}{2}\mathbf{a}_1^2$	$+\frac{1}{2}\mathbf{a}_1^2$	$(1/8^{1/2})a_1(-\mathbf{i}-\mathbf{j}+\mathbf{k}-(5^{1/2})\mathbf{l})$	$60F$ $48L$
					$\mathbf{a}_1^2$	$+\frac{1}{2}\mathbf{a}_1^2$	$(1/8^{1/2})a_1(-\mathbf{i}+\mathbf{j}-\mathbf{k}-(5^{1/2})\mathbf{l})$	$40K$ $40N$
						$\mathbf{a}_1^2$	$(1/8^{1/2})a_1(\mathbf{i}-\mathbf{j}-\mathbf{k}-(5^{1/2})\mathbf{l})$	$20T$
							α_4 lattice	

Note that when any of the operations *KLMNRST* and *Z* occur in a group the corresponding primed operations occur equally often. The primes have therefore been dispensed with in the table, following Hurley's procedure.

by centring, and the present procedure is designed to exclude the necessity of considering such cases. Sometimes it may be convenient to regard group 9 as a centring of group 11, or 16 as a centring of 12, but this is not so for counting the groups. As long as provision is made for detecting a lattice in its primitive description, further descriptions are unnecessary.

Examining first the orthogonal group number 7,

$$\begin{bmatrix} a_1^2 & 0 & 0 & 0 \\ & a_2^2 & 0 & 0 \\ & & a_3^2 & 0 \\ & & & a_4^2 \end{bmatrix},$$

where the coordinates x_1, x_2, x_3, x_4 are not connected to each other by any symmetry elements. The symmetry is seen to be

$$\begin{pmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}$$

and the following sets of equivalent positions are self-consistent (see Fig. 1).

6. Discussion of certain lattices

The majority of lattices listed are readily appreciated as orthogonal combinations of [1] and [3] or [2] and [2] lattices. Some of the remainder however merit further attention.

The metric tensor for group 16 is

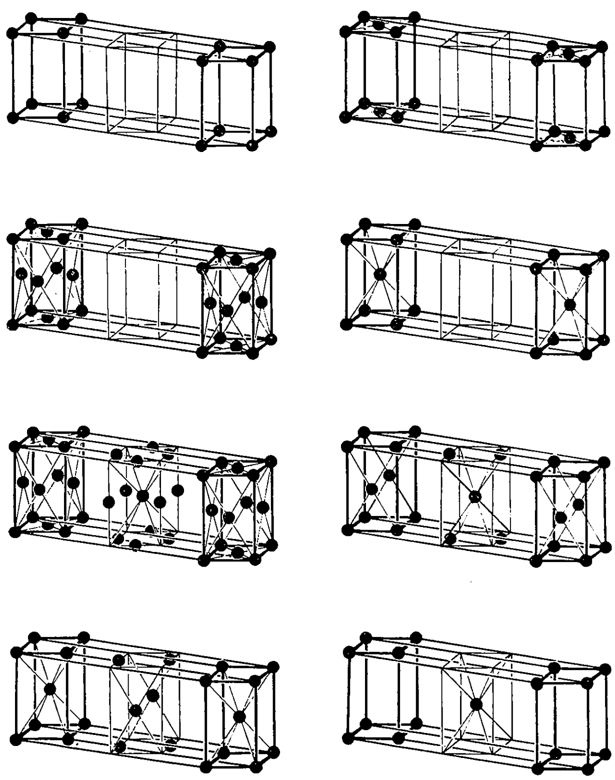

Fig. 1. The 8 orthorhombic lattices in [4].

Nomenclature of centering	P	1[2]	3[2]	1[3]	6[2]+1[4]	2[2]+1[4]	1[2]+2[3]	1[4]
Coordinates	0 0 0 0	0 0 0 0 ½ ½ 0 0	0 0 0 0 ½ ½ 0 0 ½ 0 ½ 0 0 ½ ½ 0	0 0 0 0 ½ ½ ½ 0	0 0 0 0 ½ ½ 0 0 ½ 0 ½ 0 ½ 0 0 ½ 0 ½ ½ 0 0 ½ 0 ½ 0 0 ½ ½ ½ ½ ½ ½	0 0 0 0 ½ ½ 0 0 0 0 ½ ½ ½ ½ ½ ½	0 0 0 0 ½ ½ 0 0 ½ 0 ½ ½ 0 ½ ½ ½	0 0 0 0 ½ ½ ½ ½
Number of axes with equivalent symmetry	4	2 and 2	1 and 3	1 and 3	4	2 and 2	2 and 2	4

$$a_1^2 \begin{bmatrix} 1 & \alpha & \alpha & \alpha \\ & 1 & \alpha & \alpha \\ & & 1 & \alpha \\ & & & 1 \end{bmatrix}.$$

This is obviously the [4] analogue of the rhombohedral [3] lattice R for which the metric tensor is

$$a_1^2 \begin{bmatrix} 1 & \alpha & \alpha \\ & 1 & \alpha \\ & & 1 \end{bmatrix}.$$

If the [3] lattice is transformed by the matrix

$$\begin{pmatrix} 1 & 1 & \bar{1} \\ 1 & \bar{1} & 1 \\ \bar{1} & 1 & 1 \end{pmatrix}$$

the metric tensor becomes

$$\begin{bmatrix} 3-2\alpha & 2\alpha-1 & 2\alpha-1 \\ 2\alpha-1 & 3-2\alpha & 2\alpha-1 \\ 2\alpha-1 & 2\alpha-1 & 3-2\alpha \end{bmatrix}.$$

Clearly if $\alpha=\frac{1}{2}$ this tensor becomes

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The volume ratio of 4 between the new and the original cell is given by the determinant of the transformation. Therefore the transformation of a rhombohedral lattice with $\alpha=\frac{1}{2}$ gives the cubic F lattice which has four lattice points in the unit cell, and the symmetry is increased from $\bar{3}m$ to $m3m$. The coordinates of the lattice points inside the new unit cell can be found by applying the inverse transformation

$$\begin{pmatrix} 1 & 1 & \bar{1} \\ 1 & \bar{1} & 1 \\ \bar{1} & 1 & 1 \end{pmatrix}^{-1} = \tfrac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

to simple vectors of the old lattice. Thus (100), (010), and (001) yield $(\frac{1}{2}\frac{1}{2}0)$, $(\frac{1}{2}0\frac{1}{2})$ and $(0\frac{1}{2}\frac{1}{2})$ respectively, which with (000) are the four lattice points which define the new cell. If the above procedure is repeated with $\alpha=-\frac{1}{3}$ the cubic I lattice is obtained with two lattice points in the unit cell.

If the [4] tensor above is transformed to new axes by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & \bar{1} & \bar{1} \\ 1 & \bar{1} & \bar{1} & 1 \\ \bar{1} & 1 & \bar{1} & 1 \end{pmatrix},$$

$$g_{ij}=a_1^2 \begin{bmatrix} (4+12\alpha) & 0 & 0 & 0 \\ & (4-4\alpha) & 0 & 0 \\ & & (4-4\alpha) & 0 \\ & & & (4-4\alpha) \end{bmatrix}.$$

The volume of the new cell is sixteen times that of the old but the cell is now orthogonal. The inverse transformation is

$$\tfrac{1}{4} \begin{pmatrix} 1 & 1 & 1 & \bar{1} \\ 1 & 1 & \bar{1} & 1 \\ 1 & \bar{1} & \bar{1} & \bar{1} \\ 1 & \bar{1} & 1 & 1 \end{pmatrix}$$

and hence the lattice points which fall inside the new cell are:

$$\begin{matrix} (0\;0\;0\;0) & (0\;0\;\frac{1}{2}\;\frac{1}{2}) & (0\;\frac{1}{2}\;\frac{1}{2}\;0) & (0\;\frac{1}{2}\;0\;\frac{1}{2}) \\ (\frac{1}{2}\;\frac{1}{2}\;\frac{1}{2}\;\frac{1}{2}) & (\frac{1}{2}\;\frac{1}{2}\;0\;0) & (\frac{1}{2}\;0\;0\;\frac{1}{2}) & (\frac{1}{2}\;0\;\frac{1}{2}\;0) \\ (\frac{1}{4}\;\frac{1}{4}\;\frac{1}{4}\;\frac{3}{4}) & (\frac{1}{4}\;\frac{1}{4}\;\frac{3}{4}\;\frac{1}{4}) & (\frac{1}{4}\;\frac{3}{4}\;\frac{3}{4}\;\frac{3}{4}) & (\frac{1}{4}\;\frac{3}{4}\;\frac{1}{4}\;\frac{1}{4}) \\ (\frac{3}{4}\;\frac{3}{4}\;\frac{3}{4}\;\frac{1}{4}) & (\frac{3}{4}\;\frac{3}{4}\;\frac{1}{4}\;\frac{3}{4}) & (\frac{3}{4}\;\frac{1}{4}\;\frac{1}{4}\;\frac{1}{4}) & (\frac{3}{4}\;\frac{1}{4}\;\frac{3}{4}\;\frac{3}{4}). \end{matrix}$$

This group can thus be expressed as a centering of group 12 but with the extra lattice points arranged with a symmetry lower than that of group 12. In fact as no special value of α has been assumed the symmetry must still be that of group 16.

Group 21 is the same as group 16 but with $\alpha=\frac{1}{2}$. The order of group 21 is five times that of group 16, as it has an additional symmetry element L of order 5. Group 21 is the only group to possess an element of this order. Following the above procedure it can be expressed in terms of orthogonal axes which give a metric tensor

$$a_1^2 \begin{bmatrix} 10 & 0 & 0 & 0 \\ & 2 & 0 & 0 \\ & & 2 & 0 \\ & & & 2 \end{bmatrix}.$$

The axes are therefore proportional to $\sqrt{5}$, 1, 1 and 1. This group is that of the regular pentatope α_4 (Coxeter, 1948), where the centre of the pentatope is not a lattice point. However if $\alpha=-\frac{1}{4}$ another lattice is developed which has the centre of the pentatope as a lattice point. By transforming to orthogonal axes the metric tensor becomes

$$a_1^2 \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 5 & 0 & 0 \\ & & 5 & 0 \\ & & & 5 \end{bmatrix}$$

and these axes are proportional to $1/\sqrt{5}$, 1, 1, and 1.

These lattices can be better understood by considering the vectors between the vertices of the regular α_4. The vertices are $(1, 1, 1, -1/\sqrt{5})$, $(1, -1, -1, -1/\sqrt{5})$, $(-1, 1, -1, -1, -1/\sqrt{5})$, $(-1, -1, 1, -1/\sqrt{5})$, and $(0, 0, 0, 4/\sqrt{5})$, and the origin is the centre. To form the lattice vectors the difference is taken between the coordinates of the vertices. This gives: (0, 2, 2, 0), (2, 0, 2, 0), (2, 2, 0, 0) and $(1, 1, 1, -\sqrt{5})$. Combination of these gives the lattice vectors tabulated in Table 2. A different combination gives four orthogonal vectors (4, 0, 0, 0), (0, 4, 0, 0), (0, 0, 4, 0) and $(0, 0, 0, -4\sqrt{5})$, which lead to the first metric tensor of the previous paragraph. It was there stated that the centre of the α_4 was not a lattice point. This can now be seen to be true, as no combination of the lattice vectors can give any of the vectors from the centre of the α_4 to the vertices which are given at the beginning of this paragraph. If these are now

Table 3

Lattice number (see Table 2)	Number of centred lattices	Fractional coordinates of lattice points (0000) and:	Description (number of points and dimensionality of centering)
1	0	—	—
2	1	(½ ½ 0 0)	1[2]
3	2	(½ 0 ½ 0) (½ 0 ½ 0) (0 ½ 0 ½) (½ ½ ½ ½)	1[2] 2[2]+1[4]
4	4	(½ ½ 0 0) (½ 0 ½ 0) (½ ½ ½ 0) (½ ½ 0 0) (½ 0 ½ 0) (0 ½ ½ 0)	1[2] 1[2] 1[3] 3[2]
5	0	—	—
6	1	(½ ½ ½ 0)	1[3]
7	7	(½ ½ 0 0) (½ ½ ½ 0) (½ ½ ½ ½) (½ ½ 0 0) (½ 0 ½ 0) (0 ½ ½ 0) (½ ½ 0 0) (½ 0 ½ ½) (0 ½ ½ ½) (½ ½ 0 0) (0 0 ½ ½) (½ ½ ½ ½) { (½ ½ 0 0) (½ 0 ½ 0) (½ 0 0 ½) (0 ½ ½ 0) { (0 ½ 0 ½) (0 0 ½ ½) (½ ½ ½ ½)	1[2] 1[3] 1[4] 3[2] 1[2]+2[3] 2[2]+1[4] 6[2]+1[4]
8	2	(½ 0 ½ 0) (½ 0 ½ 0) (0 ½ 0 ½) (½ ½ ½ ½)	1[2] 2[2]+1[4]
9	1	(½ ½ ½ ½)	1[4] (see Fig. 2)
10	4	(0 0 ½ ½) (½ ½ ½ 0) (½ ½ ½ ½) (0 0 ½ ½) (½ ½ ½ 0) (½ ½ 0 ½)	1[2] centred rectangle 1[3] 1[4] 1[2]+2[3]
11	1	(0 0 ½ ½)	1[2] centred rectangle
12	4	(0 ½ ½ ½) (½ ½ ½ ½) (0 ½ ½ 0) (0 ½ 0 ½) (0 0 ½ ½) (½ ½ 0 0) (½ 0 ½ 0) (½ 0 0 ½) (0 ½ ½ 0) (0 ½ 0 ½) (0 0 ½ ½) (½ ½ ½ ½)	1[3] 1[4] 3[2] 6[2]+1[4]
13	0	—	—
14	1	(½ ½ ½ ½)	1[4]
15	0	—	—
16	1	(½ ½ ½ ½)	1[4]
17	0	—	—
18	0	—	—
19	0	—	—
20	1	(½ ½ ½ ½)	1[4]
21	1	(½ ½ ½ ½)	[4] centred α_4 lattice

included as lattice vectors the centered α_4 lattice is developed. Combination of these vectors gives four orthogonal vectors: (4, 0, 0, 0), (0, 4, 0, 0), (0, 0, 4, 0) and $(0, 0, 0, 4/\sqrt{5})$ which lead to the second metric tensor of the previous paragraph. This is the only centered lattice of the pentatope and is listed in Table 3.

Hurley (1951) shows that the centered α_4 and the centered 24-cell (Coxeter, 1948) both form lattices, although some of the vertices of the α_4 have wrong coordinates in Hurley's paper. It turns out that only the primitive α_4 lattice was not considered, as the 24-cell lattice requires the centre to be a lattice point.

The vertices of a 24-cell are the mid-points of the edges of a regular β_4 (Coxeter, 1948), which is the analogue of the octahedron in [4]. The β_4 has vertices:

$$(\pm 2, 0, 0, 0),\ (0, \pm 2, 0, 0),\ (0, 0, \pm 2, 0) \text{ and } (0, 0, 0, \pm 2),$$

and therefore the 24-cell has vertices:

$$(\pm 1, \pm 1, 0, 0),\ (\pm 1, 0, \pm 1, 0),\ (\pm 1, 0, 0, \pm 1),\ (0, \pm 1, \pm 1, 0) \text{ etc.}$$

and taking vectors between the vertices such as:

$$(1, 1, 0, 0) - (0, 1, 1, 0) = (1, 0, -1, 0)\ ,$$

one of the set of vectors from the centre to a vertex is obtained. This proves that the centre of the 24-cell must be a lattice point. By a combination of vectors such as:

$$(1, 1, 0, 0)+(1, -1, 0, 0)=(2, 0, 0, 0) ,$$

the vertices of the original β_4 are seen to be lattice points. The β_4 has the same symmetry as the hypercube or γ_4 (Coxeter, 1948) which shows that the lattice of the 24-cell can be considered as a centering of the hypercubic lattice no. 20, but as the symmetry is increased it is most reasonable to list the 24-cell lattice as a new lattice type and to omit the centering 6[2]+1[4] of group 20 from Table 3.

7. Colour groups

Coloured lattices in [$n-1$] can be obtained from the normal lattices in [n] by representing fractional coordinates along one axis by a sequence of colours. If points with an x_1 coordinate 0 are represented as black then equivalent points with an x_1 coordinate of $\frac{1}{2}$ can be represented as white. Equivalent points on p different levels will require p different colours when the original lattice is represented in one less dimension. It must be stressed that such a coloured lattice is not a complete representation of the generating lattice, unless there is the tacit assumption that the axis supressed and represented by colours is orthogonal to all the others, as the angular relationship of the axis supressed to the others is lost. Several coloured lattices may be obtained from a single plain lattice by projecting in different directions, and several plain lattices may give the same coloured lattice. The 14 [3] space lattices give 5 black and white [2] lattices and, for example, the rhombohedral lattice if projected along its triad axis requires 3 colours as it has equivalent points on three levels. This gives, in fact, the arrangement of coloured points on the screen of a colour television set.

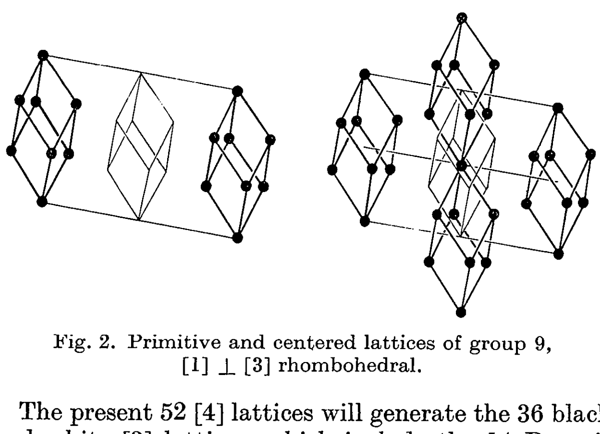

Fig. 2. Primitive and centered lattices of group 9, [1] ⊥ [3] rhombohedral.

The present 52 [4] lattices will generate the 36 black and white [3] lattices, which include the 14 Bravais lattices, and certain multicoloured lattices. Fig. 2 shows the centred lattice of group 9 and the primitive lattice. If the centred lattice points are coloured white and projected into the three dimensions of the rhombohedral lattice, the two-colour rhombohedral lattice is produced.

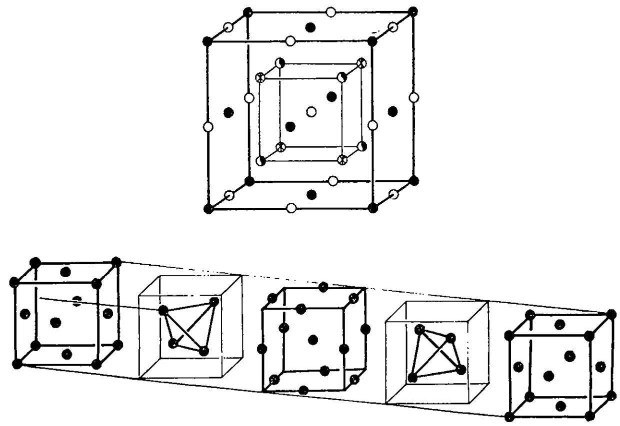

Fig. 3. A four colour [3] cubic cell and its relation to the lattice of group 16, the [4] analogue of the [3] rhombohedral lattice.

The most interesting of the multicoloured lattices comes from lattice 16 discussed in the previous section. If the axis corresponding to the vector **1** (see Table 2) is made the colour axis, the cell can be reduced to [3] as shown in Fig. 3. The unit cell contains eight cells of the body-centred cubic lattice divided into four face-centred cubic lattices. This structure resembles one of the CsCl or NaTl superstructures, for example ordered Fe_3Al (Wells, 1962).

8. Conclusions

In [4] it has already been shown that there are 24 symmetry operations not involving translations and 222 crystallographic point groups. It is here shown that there are 52 translation lattices, 21 primitive and 31 centered, although as the method of derivation has been enumerative omission is possible.

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